Lin, P. J. , Tsai, W. H. (2012). Future teachers' proof of universal and existential statements. *Paper presented at the 12th International Congress on Mathematical Education (ICME12)*. TSG 23. July 8–15, 2012, COEX, Seoul, Korea.

FUTURE TEACHERS' PROOF OF UNIVERSAL AND EXISTENTIAL STATEMENTS

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The study is intended to explore how the FTs verifying the six given universal and existential statements. Eight FTs consisting of 4 with mathematics background and 4 without mathematics background participated in the study. The selection of the eight FTs' beliefs was based on the score of survey of beliefs of nature of mathematics, mathematics learning, and mathematicsteaching. The participants were given six statements including three universal and three existential statements. The result indicates that the universal statements were easier for the FTs than existential statements. The FTs with mathematics background are more flexible and successful intransition of various expressions and used advanced algebra to give the valid arguments. Some of FTs without mathematicsbackground believed that finding a finite number of discrete cases can prove a universal statement.

Key words: future teachers, universal statement, existential statement, proof.

THE IMPORTANCE OF KNOWLEDGE ABOUT PROOF

Proving has been considered central to the mathematics learning. Proving can be a vehicle for deep learning in all content areas (Reid &Knipping, 2010). Mathematics education reforms highlight the importance of reasoning and proof in school mathematics (NCTM, 2000). Mathematical instruction emphasizes that students should have the opportunity of exploring, investigating, conjecturing, explaining, and justifying mathematics (NCTM, 2000). Engaging in proving can support students to explore why things work in mathematics, so that proof is a vehicle to enhancestudents' understanding of mathematical concepts and promote their mathematical proficiency and reasoning.

If mathematics teachers have an inadequate understanding of proof, it is not surprised that students also have similar misunderstanding. Unless teachers have good understanding of proof, we cannot expect that they will be able to effectively promote proving among their students. Thus, to foster students' engaging in proving activity, it is also important for teacher to have understanding of proofs themselves. The purpose of the study is to explore how future teachers make a judgment and prove the given six statements.

THEORETICAL FRAMEWORK

The definition of proof has different meaning from different criteria. For example, from perspective of mathematics as a discipline, Stylianides& Ball (2008) break each mathematical argument into three major components: 1) the set of accepted statements, 2) the modes of argumentation, 3) the representation of modes of argument. Proofs are set of true statements, valid modes of argumentation, and appropriate argument representation. Valid modes of argumentation are deduction, induction, contraposition equivalence rule, and the construction of counterexamples.

Regarding the consideration of students as mathematical learner, Balacheff (1988) proposes four levels of proof according to students' work rather than by a strict mathematical meaning. Naïve empiricism at level 1 refers to a conjecture to be true by verifying some discrete cases. The crucial experiment at level 2 is that testing a conjecture in a special or extreme case. The distinction of level 2 is from level 1 in that students are aware of the problem of generality. The first two levels is similar the empirical induction from a finite number of discrete cases suggested by Canadas, Deulofeu, Figueiras, Reid, &Yevdokimov, (2007). The generic example at level 3 is to show the truth by manipulating an object that is used as a representative of all similar objects. The proof is indicated by the effect of the operations. The proof at level 4, the thought experiment, is indicated by looking at the properties of the objects, not the effects of operations.

Balacheff (1988) stresses that the hierarchy of proof levels deals with the method that students use, not with how correct or successful the students are. It could be happened that a student's proof is classified at level 4, but it is not fully correct or complete. The three components Stylianides& Ball (2008) proposed as are utilized as the framework of the study for analyzing the way of future teachers' proof for six given statements. Balacheff's proof of levels is the basis of identifying the modes of argument used by the FTs involving in the study.

THE EMPIRICALSTUDIES OF PROOF

Reid and Knipping (2010) review a great number of the studies on proof across countries and across age groups. The subjects more are focused on secondary school students, and university students. Future elementary school teachers, inservice upper elementary and secondary school teachers are included. It is found that many students accept examples as verification and do not accept counterexamples as refutation. Galbraith (1981) reports that some of the 12-17 years old Australian students agree that a single counterexample was insufficient to refute a general statement.

Ko (2010) also review a number of studies on elementary and secondary mathematics teachers' conceptions of proof around the world. Future teachers' have the difficulty with proof by mathematical induction (Stylianides et al., 2007). Barkai, Tsamir, Tirosh, and Dreyfus (2002) suggest that inservice upper elementary school teachers in Israel do not accept a counterexample to a universal statement as a valid proof, such as the statement"The sum of any four consecutive integers is divisible by four". They do not believe that a single counterexample is sufficient to refute a universal statement.

Barkai, et al. (2002) suggest that the existence of something for inservice teachers is easier than proving the non-existence of something. For examples, 64% of the 27 inservice upper elementary school teachers in Israel can prove the statement "There exist three consecutive integers whose sum is divisible by six", while only 23% of the teachers can prove the statement "There exist four consecutive integers whose sum is divisible by four". The latter statement requires a deductive approach because they assert that something is true for an infinite number of cases. On the contrary, the statement that can be proved by using examples is much easier for inservice teachers. For example, the statement "There exist five consecutive integers whose sum is divisible by five" that can be proved by use some examples.

The previous studies did not focus the factors of the mathematics background and beliefs of mathematics teaching and learning on the effect of proof, instead, focusing on the subject at different grade levels, such as lower and upper secondary school students, undergraduate students, future students, and inservice students. Proof requires rigor processes and valid statements. It is trained in mathematics courses. Thus, we assume that one who has mathematics ground is more successful than one who has not received mathematics training. Thus, the purpose of the study is to explore how the future teachers with/without mathematics background verify the universal and existential statements.

RESEARCHMETHOD

Participants

Eight future teachers (FTs) who were taking the Mathematics Teaching Method course of the teacher education program participated in the study. The participants were intentionally selected by the instructor of the course from an entire class with 35 FTs enrolling in the course. The reasons of selection were based on: 1) the diversity of background (mathematics vs. non-mathematics background), 2) beliefs of the nature of mathematics, mathematics learning, and mathematics teaching.

The 35 FTs were conducted by a questionnaire with 60 items of teachers' beliefs. The beliefs scales include questions in three areas: belief about the nature of mathematics, beliefs about learning mathematics, and beliefs about mathematics teaching. The items of the nature of mathematics are to explore how FTs perceive mathematics as a set of rules or inquiry. The items of beliefs about learning mathematics are to explore how FTs emphasize on active learning or teacher direction. Beliefs about mathematics teaching include the emphasis of results or processes.

Eight FTs were selected from the 32 FTs based on the scores of the survey. Four FTs are from mathematics background. Two (T1 and T2) of them are studying in master program and two (T3 and T4) of them are undergraduate students. Four FTs (T5, T6, T7, and T8) do not have mathematics background. T5 are graduate students and T6, T7, and T8 are undergraduate students. The background of each FT is summarized as Table 1.

T1, T2, T4, and T6 have similar beliefs about the nature of mathematics, mathematics learning, and mathematics teaching. They view mathematics as the process of inquiry instead of as a set of rules; Students should be responsible for their own learning instead of are instilled. Their views of mathematics teaching is that teaching should emphasize on the process of problem solving instead of the result only. Conversely, the beliefs of T3, T5, T7, and T8 about mathematicslearning and teaching are toward to the traditionalperspective of mathematicslearning and teaching. Each FT's belief is summarized as Table 1.

Table 1: Background of the participants

Background			T1	T2	T3	T4	T5	T6	T7	T8
Degree	Master	Math	●	●						
		Non-math					●			
	Undergrad.	Math			●	●				
		Non-math						●	●	●
Beliefs about	Nature of mathematics	Inquiry	●	●		●		●		
		Rules			○		○		○	○
	Mathematics learning	Active learning	●	●		●		●		
		Passive learning			○		○		○	○
	Mathematics teaching	Process	●	●		●		●		
		Results			○		○		○	○

Instrument

The eight FTs were asked to judge whether six statements were true or false, and to provide the justification for the statements. The statements, three true and three false, were related to divisibility of sums of consecutive numbers. The instrument includes six statements as follows.

1. *The sum of any four consecutive integers is divisible by two.*
2. *The sum of any four consecutive integers is divisible by tree.*
3. *The sum of any four consecutive integers is divisible by four.*
4. *There exist four executive integers whose sum is divisible by two.*

5. There exist four executive integers whose sum is divisible by three.

6. There exist four executive integers whose sum is divisible by four.

Three were universal statements and three were existential statements. Table 2 summarizes the type and truth value of the six statements.

Table 2: The type and truth value of the six statements.

Statement	#1	#2	#3	#4	#5	#6
Type	Universal	Universal	Universal	Existential	Existential	Existential
Truth value	True	False	False	True	True	False
True for what set?	All n	Some n	All n	All n	Some n	No n

Data Collection and Analysis

The six statements were put in one card. Each FT was given a card to write the answers on blank A4 sheets. They were encouraged to judge the true value of each statement and write a proof for convincing their friends. Each FT was conducted the test in a classroom lasting for an hour.

The judgment of true value for each statement was first summarized in a table. The process of proofs for their judgment were then cut and pasted altogether by each item from their answer sheets. The data collected were analyzed by comparing and contrasting in accordance with the three aspects of an argument: statements, modes of statements, and representation of statements (Stylianides & Ball, 2008).

RESULTS

FTs'Judgment of True Value of Six Statements

All the FTs are successful in making the judgment for the statements #1, #2, #3, and #4, while those who (T7 and T8) do not have mathematics background and have their beliefs about traditional mathematics learning and teaching had the difficulty with judging the true value of the statements #5 and #6, as indicated in Table 3. Of those who made correct judgments of these statements, almost all could give correct proofs. The result indicates that the universal statements were easier for the FTs than existential statements.

Table 3: FTs'judgment of the true value for the six statements.

Statement	#1	#2	#3	#4	#5	#6
Type	Universal	Universal	Universal	Existential	Existential	Existential
Truth value	True	False	False	True	True	False
True for what set?	All n	Some n	All n	All n	Some n	No n
Correct judgment	T1, T2, T3, T4, T5, T6, T7, T8	T1, T2, T3, T4, T5, T6, T7, T8	T1, T2, T3, T4, T5, T6, T7, T8	T1, T2, T3, T4, T5, T6, T7, T8	T1, T2, T3, T4, T6	T1, T2, T3, T5, T6,

FTs' Proofs of the Six Statements

Statement #1 and #4

There were two modes of arguments used by the eight FTs for statement #1 and statement #4, seen as Table 4. At naïve empiricism level, only two FTs without mathematics background used one or three cases to illustrate their proofs. The rest of the 6 FTs' proofs were at the thought experiment. Within this level, two mathematics concepts were used: One is by using the properties of the sum of two odds and the two evens. The other is proved by the use of algebraic thinking. Table 4 displays that three representations of argument used by the FTs were: number sentences, word expressions, and algebraic expressions.

Table 4: FTs' proof for the statement #1 and #4.

1. The sum of any four consecutive integers is divisible by two.

4. There exist four executive integers whose sum is divisible by two.

A set of statements		Modes of argument	Representations of argument
(1) 1+2+3+4=10÷2=5 (2) 2+3+4+5=14÷2=7 3+4+5+6=18÷2=9 4+5+6+7=22÷2=11	T5 T7	Naïve empiricism: Empirical induction from one or three cases	Number sentences
(1) There are 2 evens and 2 odds in the four consecutive numbers. The sums of 2 odds become even number. The sums of 2 even numbers are still even. So that the sums of the four consecutive numbers are even numbers. (2) Let the four consecutive numbers are: n, n+1, n+2, n+3.	T2, T4	The thought experiment: The proof is by looking at the properties odd+odd= even &even+even=even Even number can be divided by 2.	Word descriptions
n+n+1+n+2+n+3=	T1,T8	The sum of the four consecutive four numbers is	Algebraic expressions

4n+6=2(2n+3) Or∵2 | 4n & 2 | 6 ∴2| 4n+6	, T6 T3		represented as 4n+6=2(2n+3) Here 2 is a factor of 2(2n+3)

Statement #2 and #5

The two FTs F2 and F4 convinced others by using the properties of the sum of odds and evens in statement #1. It is hardly to use the same method for successfully proving statement #2 and #5. T4 without mathematics background verified the statement #2 by using a case, while T2 with mathematics background promoted by using the judgment of algebraic expression 4n+6 for statement #2. T4 verified the statement #2 with universal type by finding a counterexample and the statement # 5 with existential type for using a case. T5's and T7's proofs for statement #2 still utilized empirical induction with finite number of discrete cases same as the statement #1. Likewise, the four FTs, T1, T3, T6, and T8, still used the higher level of proof for statement #2 same as statement #1 by looking at the sum 4n+6 is a multiple of 3 whether n is a multiple of 3 or not.

Table 5: FTs' proof for the statement #2 and # 5.

2.The sum of any four consecutive integers is divisible by three.
5.There exist four executive integers whose sum is divisible by three.

A set of statements		Modes of argument	Representations of argument
(1) 2+3+4+5=14 is not divided by 3 (2) 2+3+4+5=14÷3=4….2 3+4+5+6=18÷3=6 4+5+6+7=22÷3=7…1	T4,T7 T5	<u>The thought experiment:</u> (1) By using one or more counterexamples.	Number sentences
Let the four consecutive numbers are: n, n+1, n+2, n+3. n+n+1+n+2+n+3=4n+6 is divided by 3 if n is a multiple of 3. or =3(n+2)+n orn+n+1+n+2+n+3=4n+6=2(2n+3) or∵3 | 6 but 4n+3 is not always multiples of 3 ∴3| 4n+6 is not always true	T2, T6 T1, T8, T3	(2) The proof is by looking at the 4n+6 is a multiple of 3 determined by whether n is a multiple of 3 or not.	Algebraic expressions

Statement #3and #6

For proving the statements #1, #2, and #3, T1, a graduate student of mathematics department, flexibly expressed 4n+6 with various algebraic expressions for its purpose, for instance, 4n+6=2(2n+3) for judging whether it is a multiple of 2; 4n+6=3(n+2)+n for judging whether it is a multiple of 3; and 4n+6=4(n+1)+2 for judging whether it is a multiple of 4. Likewise, T3, a graduate student of mathematics department, consistently used the same method for the statements #1, #2, and #3 to judge if 4n+6 is divisible by 2, or 3, or 4, determined by two terms, 4n and 6 if both are divisible by 2, or 3, or 4.

Comparing to the arguments for proving the statement #2 and #5, all FTs used the same statements for proving statement #3 and #6, as seen in Table 6. Only T2 utilized a counterexample to verify the statement #3 is a false statement. T1, T3, T6, and T8 successfullyverified the statement #3 by using logic deduction.

For the statement #6, it is false statement. It never exists that four executive integers whose sum is divisible by four. T4 and T7 were still intended to find one or three cases to illustrate it. Finally, they are failed to make a judgment for statement #6. T8 used an algebraic expression to represent the sum of any four consecutive integers as 4n+6=2(2n+3), but she still failed to prove if it is a multiple of 3 or 4, it leads to her incorrect judgment. Moreover, T8 without math background and her traditional views of mathematics teaching and learning consistently expressed the four consecutive integers as 4n+6=2(2n+3) for the six statements, but she did not give any illustrations.

Table 6: FTs' proof for the statement #3 and #6.

3.The sum of any four consecutive integers is divisible by four.
6. There exist four executive integers whose sum is divisible by four.

A set of statements		Modes of argument	Representations of argument
(1) 2+3+4+5=14 is not divided by 4 (2) 1+2+3+4=10÷2=5/2 (3) 2+3+4+5=14÷4=3….2 3+4+5+6=18÷4=4…2 4+5+6+7=22÷4=5…2	T4 T7 T5	The thought experiment: (1) Use one or more counterexample.	Number sentences
Let the four consecutive numbers are: n, n+1, n+2, n+3. n+n+1+n+2+n+3=4n+6 if n=1, then it is not true. or 4n+6 is not divided by 4 or =4(n+1)+2 orn+n+1+n+2+n+3=4n+6=2(2n+3) or∵4 l4n but 6 is not divided by 4 ∴4l 4n+6 is false.	T2, T6 T1 T8, T3	(2) The proof is by looking at the 4n+6 is not a multiple of 4.	Algebraic expressions

CONCLUSIONS

The result indicates that mathematics background and beliefs of mathematics teaching and learning seem to be two factors of affecting FTs' proof of the six universal and existential statements. Moreover, the FTs with mathematics background were more successful and flexible in changing the term 4n+6 into the various expressions as 3(n+2)+n , 4(n+1)+2, and 2(2n+3) according to the need of proving each statement, such as T1. T2 is also flexible to use various modes of argument to prove the six statements. She used the properties of the sum of odd and even numbers for verifying the universal statement #1, but she switched to look at the 4n+6 is a multiple of 3 determined by whether n is a multiple of 3 or not for verifying the universal statement #2. For verifying the false of statement #3, T2 used the counterexample as the mode of argument. Likewise, T3 with mathematics background, coherently and successfully used the advanced algebra knowledge if m l n (m is a factor of n) & p l n (p is a factor of n) , then (m+p) l n (m+p is a factor of n) to judge if 4n+6 is divisible by 2, or 3, or 4.

The FTs without mathematics background believe that finding a finite number of discrete cases can be a proof for a universal statement. For T7 and T5 consistently provided one or three cases to prove true universal statement and false existential statements. This indicates that they are not knowledgeable with the proofs of universal and existential statements.

T6 without mathematics background but with constructivists' perspective of mathematics learning, she is successful in proving the six statements. T7 and T8 are not studying in mathematics department. They did not succeed in proving the existential statements #5 and #6.The result also indicates that the universal statements were easier for the FTs than existential statements. The result is consistent with the study of Barkai, et al. (2002) for inservice teachers. For the six statements, algebra expression is more often to be used as the expression of the arguments than the word expression.

Acknowledgement

The paper is based on the data as part of theresearch project of "The study of teacher professional development on teacher educator's support for assisting teachers in helping students' mathematical learning" Granted by National Science Council, Contract No.：NSC 99-2511-S-134 -005 -MY3.

References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers, and children* (pp. 216-235). London: Hoddler and Stoughton.

Barkai, R., Tsamir, P., Tirosh, D., & Dreyfus, T. (2002). Proving or refuting arithmetic claims: The case of elementary school teachers. In A. Cockburn & E. Nardi (Eds.), Proceedings of the *26th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.57-64). Norwich, UK.

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education*, *8*:171-191.

Canadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., & Yevdokimov, O. (2007). The conjecturing process: Perspectives in theory and implications in practice. *Journal of Teaching and Learning, 5*(1), 55-72.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., &Loef, M. (1989).Using knowledge of children’s mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*, 499-531..

Galbraith, P. L. (1981). Aspects of proving: A clinical investigation of process. *Educational Studies in Mathematics, 12*(1), 1-28.

National Council of Teachers of Mathematics (2000).*The principles and standards for school mathematics*. Reston, VA.: National Council of Teachers of Mathematics.

Ko, Y. Y. (2010). Mathematics teachers’ conceptions of proof: implications for educational research. *International Journal of Science and Mathematics Education, 8*, 1109-1129.

Reid, D. &Knipping, C. (2010).*Proof in mathematics education: research, learning, and teaching*. The Netherlands: Sense publisher.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Stylianides, G. J. ,Stylianides, A. J. &Philippou, G. N. (2007). Preservice teachers’ knowledge of proof by mathematical induction. *Journal of Mathematics Teachers Education*, *10*, 145-166.

Stylianides, A. J. ,& Ball, D. L. (2008). Understanding and describing mathematical knowledgefor teaching: knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education 11*:307–332