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FIFTH GRADERS' MATHEMATICS PROOFS IN CLASSROOM CONTEXTS

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The study intends to identify what mathematical proof looks like in primary classroom contexts. It describes the three components of mathematical proof observed in the episode of fraction comparisons in a grade 5 class. Students' written work collected from classroom, videos and transcriptions of classroom observations were the main data gathered in the study. Empirical induction from finite number of discrete cases for verifying a conjecture to be true and offering more than one counterexamples rather than only one counterexample for refuting a false proposition were two common forms of reasoning used by the fifth graders. Teacher's authority was another way of proof in the fifth grade classroom. The fifth graders accepted Q →~P as the negative of a proposition ~(P→Q).

INTRODUCATION

Mathematical reasoning is a focus of mathematical instruction that involves students in exploring, investigating, conjecturing, explaining, and justifying mathematics (NCTM, 2000). The document states that teachers should provide students from K to12 grade level with the opportunities "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematics arguments and proofs, and select and use various methods of proof."

Proof is a vehicle to enhance students' understanding of mathematics concepts and promote mathematical proficiency and reasoning (Hanna, 2000). Proving is an important means of exploring in mathematics. Research shows that engagement in proving can support students to explore why things work in mathematics and explain their disagreements in meaningful ways, thus providing them with a solid basis for conceptual understanding (Stylianides, 2007).

Researchers draw more attentions on mathematics reasoning, proof and proving, and conjecturing. For example, international conference ICMI 19 study has a special issue of proof and proving in mathematics education (Lin, Hsieh, Hanna, & de Villiers, 2009). Some studies focus on students' verification with mathematics (Reid, 2002), and emphasize on the formal and informal aspects of proof (Canadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007). Some focus on the roles that proof might take in the mathematics curriculum if it is to be taught effectively. Teaching that supports proof (Stylianides & Ball, 2008) and teachers' conceptions of proof that influence the opportunities they create for students to engage in proving are also addressed (Ko, 2010). These increasing studies of proofs and proving has been limited to high school geometry. Additionally, existing empirical studies suggest that some high school students or undergraduates have difficulty with mathematical proof in school mathematics (Ko, 2010). The above discussion indicates that students should have early and appropriate opportunities to incorporate mathematical concepts and proof into their mathematical learning. Taking this consideration, this study encouraged elementary school teachers to provide students the opportunity to engage in proving activities. The proofs and proving were a new experience for primary school students and teachers involving in the study. Thus, this study is intended to identify what mathematical proof looks like exploring in a primary classroom context.

CONCEPTUAL FRAMEWORK

Stylianides and Ball (2008) distinguish the definition of proof from two considerations: mathematics as a discipline and students as mathematical learners. From mathematics as a discipline, a proof is a mathematics argument that requires three components: a set of true statements, valid modes of argumentation, and appropriate modes of argument representation (Stylianides, 2007). According to the criteria, empirical arguments cannot count as proof at any level of schooling, because empirical arguments utilize invalid modes of argument by promoting acceptance of mathematical claims based on incomplete evidence (Stylianides and Ball, 2008).

Regarding the consideration of students as mathematical learners, proof is defined as a set of accepted statements, known modes of argumentation, and accessible modes of argument representation to a classroom community. According to the criteria, students' proofs can be distinguished into four levels (Balacheff, 1988). *Naïve empiricism* at Level 1 refers that a conjecture is true after verifying some cases. *The crucial experiment* at Level 2 involves testing a conjecture by using a special or extreme case. The distinction from Level 1 is in that the students are aware of the generality. *The generic example* at Level 3 refers to showing the truth by manipulating a representative objective. The proof is indicated by the effect of operations. Students' poof at Level 4 *thought experiment* is indicated by looking at the properties of the objectives, not at the effects of operation on the objects. The conceptual framework of this study takes the consideration of students as mathematical learners. Thus, students' proofs consist of the three components: the set of accepted statements, understandable modes of argumentation, and accessible modes of argument representations, which do not focus on correctness or success that students produced at primary mathematics classroom will be identified.

RESEARCH METHOD

The proving activity described occurred in the grade 5 classroom of Lu-Lu, a teacher who participated in the first year of a three-year project that is designed to help teachers to create conjecturing tasks for engaging students in the activity of proving. Hence, the design of conjecturing tasks is new learning for Lu-Lu. Help students approaching to do valid proofs is a new experience for Lu-Lu, too.

The conjecturing task being explored was that "*2/4 is greater than 1/3, 4/5 is greater than 3/4. The teacher, Lu-Lu, finds out that both denominator and numerator of the 2/4 are greater than those of 4/5, respectively. Thus, Lu-Lu concludes: To compare any two fractions, if both denominator and numerator of the one fraction are greater than* *those of the other fraction, then the one fraction is greater than the other fraction. Do you agree? Why? Write down your reasons.*"

The conjecturing task being explored is to ask students to judge and verify. Lu-Lu stated that she would rather give fifth grade students two cases in advance to assist them in understanding the verbal statement than give them the proposition directly.

29 students in the class were grouped heterogeneously in groups of 4 or 5. When given the problem, students first worked independently and jotted down their solutions; then they came together in groups to compare their solutions, and last they reported to the whole class. The students were videotaped throughout their group and whole class discussion. Each student's written work was collected.

The first part in the result session was aligned to the Stylianides and Ball's (2008) three components of proof by using students' written work, videotapes of classroom observation, and transcriptions of discussion. In accordance with the three components of the proofs, the analysis of students' written solutions were first split into two piles and then a pile is assigned to each group of the two groups consisting of six school teachers studying in graduate program. Afterwards, they took turns to review the other pile for increasing the validity and reliability of analysis. The second part of the result session was from a small sampling from the whole class in order to illustrate a particular type of proving.

RESULTS

The Set of Statements Accepted by the Fifth Graders

After the task being explored, 23 (79%) students made incorrect judgement by accepting the teacher's conclusion, while only 6 (26%) students were correct by rejecting the conjecture; out of the 6 students, 3 (13%) students verified successfully.

The set of statements referred to the statements accepted by the classroom community. Replicating the teacher's statement was the most common statements accepted by those who were in favour of the conjecture. 20 students' set of arguments were *"if both denominator and numerator of the one fraction are greater than those of the other fraction respectively, then the one fraction is greater than the other fraction."*.

Once they figured out the new cases, the equivalence of fractions regularly became as the part of their arguments consisting of a set of statements. For instance, Janice and Krin, two of the students, successfully showed their accepted statements by the classroom community as follows.

Janice's statements:

$\frac{2}{100} < \frac{1}{2}$, since $\frac{1}{2} = \frac{50}{100}$, but here 1 is not greater than 2 and 2 is not greater than 100.

$\frac{2}{1000} < \frac{1}{2}$, since $\frac{1}{2} = \frac{500}{1000}$, but here 1 is not greater than 2 and 2 is not greater than 1000.

$\frac{12}{1000} < \frac{1}{2}$, since $\frac{1}{2} = \frac{500}{1000}$, but here 1 is not greater than 12 and 2 is not greater than 1000.

So that, I did not agree with the teacher's saying.

Krin's statements:

$\frac{3}{6} = \frac{1}{2}$, here 3 is greater than 1 and 6 is greater than 2, but 3/6 is not greater than 1/2

$\frac{2}{2} = \frac{3}{3}$, here 2 is smaller than 3 and 2 is smaller than 3, but 2/2 is not smaller than 3/3.

Thus, I did not agree with what the teacher said.
The Janice's and Krin's accepted statements further indicated that the equivalence of 1/2 and 1 became mathematical cognitive tools as part of the counterexamples.

Modes of Argument Known by the Fifth Graders

Overall, the modes of argument the fifth graders regularly utilized were empirical induction from finite number of discrete cases with one, two, or three for accepting a proposition and offering one more counterexamples for refuting a false proposition.

Furthermore, one or two cases for empirical induction was from their teacher offering in the task. For instance, of the 23 students' incorrect conjecture, 20 students' verifications were based on the two cases (2/4 > 1/3 and 4/5 > 3/4) given by the teacher, as shown in Table 1. In addition, 3 students argued that the conjecture is true after verifying three cases (e.g., 2/4>1/3, 4/5>3/4, 7/8> 6/7).

For the 6 students' correct judgement, they used counterexamples to refute a false proposition. Only 3 students verified successfully by utilizing a single counterexample to refute it, while 3 students did not agree that a single counterexample is sufficient to refute a false proposition. Thus, they offered more than one counterexamples (e.g., 2/4 is not greater than 1/2; 2/6 is not greater than1/2) to refute the proposition. Moreover, 1/2 and 1 as the referent points while comparing with the other fraction were more likely successful when finding out a counterexample.

Argument Representations Accessible by the Fifth Graders

Modes of argument representations were the forms of expression for communicating with the classroom community. For this conjecture task involving fractions, word description was the most popular form of argument used by the fifth graders. As shown in Table 1. 22 (85%) students convinced others by using word descriptions. These students utilized the two cases ($\frac{2}{4}>\frac{1}{3}$, $\frac{4}{5} > \frac{3}{4}$) given by the teacher to look for a pattern as follows. (T: the teacher, Yi: a student)

16 T: Why do you agree with the statements I proposed?

17 Yi: Because for $\frac{2}{4}>\frac{1}{3}$, the denominator 4 of $\frac{2}{4}$ is greater than 3 of $\frac{1}{3}$, the numerator 2 of $\frac{2}{4}$ is greater than 1 of $\frac{1}{3}$. Likewise, for $\frac{4}{5} > \frac{3}{4}$, the denominator 5 of $\frac{4}{5}$ is greater than 4 of $\frac{3}{4}$, the numerator 4 of $\frac{4}{5}$ is greater than 4 of $\frac{3}{4}$. Thus, I believe it is true.

Within the word descriptions for convincing others, 7 students combined the written words with pictorial representation and 2 students combined the words with symbols. In other words, 13 students used word descriptions only to explain the conjecture to be convinced. Additionally, 13 students also frequently used pictorial representation combining with either word descriptions or symbolic representations to explain the comparison of the two pairs of fraction given by the teacher.

	Set of statements		Modes of argument		Argument representations	
Correct judgement	Offering new cases	#	Counterexamples	#		#
	◦ $\frac{1}{2}>\frac{2}{6}$	1	◦ One case	3	◦ word descript.	1
	◦ $\frac{2}{2}>\frac{3}{4}$	1			◦ Pictorial	0
	◦ $\frac{1}{2}=\frac{2}{4}$	1			◦ Symbolic	0
	◦ $\frac{3}{6}=\frac{1}{2}, \frac{2}{2}=\frac{3}{3}$	2	◦ Two cases	2	◦ Word+Picto.	3
	◦ $\frac{2}{4}=\frac{1}{2}, \frac{2}{6}<\frac{1}{2}$				◦ Picto.+Symb	1
	$\frac{2}{1000}<\frac{1}{2}, \frac{12}{1000}<\frac{1}{2}, \frac{2}{100}<\frac{1}{2}$	1	◦ Three cases	1	◦ Word+Symb.	1
Incorrect judgement	Using teacher's giving cases	#	Empirical induction from	#	Word descriptions	12
	--	--	◦ One case	0	Pictorial	0
	$\frac{2}{4}>\frac{1}{3}, \frac{4}{5}>\frac{3}{4}$	20	◦ Two cases	20	Symbolic	1
	Offering new cases		◦ More than two cases	3	Word+Picto.	4
	◦ $\frac{2}{4}>\frac{1}{3}, \frac{4}{5}>\frac{3}{4}, \frac{7}{8}>\frac{6}{7}$	1			Picto.+Symb.	5
	◦ $\frac{4}{6}>\frac{1}{2}, \frac{6}{8}>\frac{2}{4}, \frac{4}{8}>\frac{1}{4}$	1			Word+Symb.	1
	◦ $\frac{8}{10}>\frac{3}{5}, \frac{4}{5}>\frac{9}{10}, \frac{9900}{10000}>\frac{9800}{9900}$	1				

#: the number of students

Table 1: Fifth graders' proofs of the given conjecturing task.

Many Fifth Graders Accepted the "~(P→Q) is (Q→~P)

During engaging the proving activity, Wei, one of the students who did not agree with the statements the teacher proposed, revised the original proposition to "*if the differences of denominators and numerators of two fractions are equal to 1,* (P), *then the one fraction with greater denominator or numerator is greater than the other fraction.*"(Q), represented as P→Q. The teacher stated that she was not sure at that moment whether the proposition is true when we discussed right after the classroom observation. Immediately, the teacher invited all of the students to judge and justify if Wei's statements is true or false.

24 of 29 students engaged in previous activity of proving realized that to verify the false proposition is to find counterexamples. Hence, they tried hard to find counterexamples to verify the false proposition. 12 of them did not recognize that a

counterexample is sufficient to verify a false proposition. Most of the students attempting to figure out the counterexamples accepted that the negative of the false proposition ($\sim(P \rightarrow Q)$) is $Q \rightarrow \sim P$ instead of $P \wedge \sim Q$. That is, most of the students accepted that the negative of the proposition "*if the differences of denominators and numerators of two fractions are equal to 1*, (P), *then the one fraction with greater denominator or numerator is greater than the other fraction*."(Q)," became as "*if one fraction is greater than the other (Q), then the differences of the denominators and numerators of two fractions respectively are not always equal to 1 (~P).*" Four students, S3, S6, S17, and S29, displayed their statements on their worksheets as follows.

S3: The counterexamples are 8/10>3/5, 6/9> 1/3, but the differences of denominator and numerator 8, 3 and 10, 5 of one pair (8/10, 3/5) are not equal to 1. Likewise, 6-1=5, 9-3=6 in the other pair, both 5 and 6 are not equal to 1, either.

S6: 4/6>1/2, 6/8>2/4, 4/8>1/4, but the difference of denominator and numerator in each pair of fractions is not equal to 1. 4-1=3 , 6-2=4; 6-2=4 , 8-4=4; and 4-1=3 , 8-4=4.

S17: 2/2=3/3. The difference of denominators of the two fractions is equal to 1, and the difference of numerators of the two fractions is equal to 1, too. But 3/3 is not greater than 2/2.

S29: 99/100<98/99. The difference of denominators of two fractions, 100-99=1, is equal to 1. The difference of numerators of two fractions, 99-98=1, is equal to 1. But 99/100 is not greater than 98/99.

The common error of the students S3, S6, and S29, for verifying the false proposition displayed that the negative of "if P, then Q", i.e., ~(if P, then Q) is "if Q, then ~P". That is, given two pairs of fraction comparisons, such as S3's two pairs (8/10>3/5, 6/9> 1/3), they do not satisfy the condition: the differences between two denominators and two numerators of the pair of fractions are equal to 1. The episode further showed that fifth graders' failed in refuting a false proposition in that they accepted "$\sim(P \rightarrow Q) \equiv Q \rightarrow \sim P$".

Furthermore, S29's statements revealed that she did not have a correct knowledge of comparing two fractions with unlike denominators. The result also indicted that without solid mathematical concepts, it is impossible for students to engage a valid mathematical proof.

For S17's arguments, he perceived with finding a counterexample to refute the conjecture. That is, an existed a case, 3/3 is not greater than 2/2, but it does satisfy the condition: the difference (3-2=1) of two denominators and the difference (3-2=1) of two numerators in these two fractions are equal to 1. Thus, this counterexample resulted in some of the students in the class to revise the proposition proposed by Wei. They further stated that they must be proper fractions.

CONCLUSION AND DISCUSSION

Relying on the three components of mathematical arguments, the first component of proof suggested by Stylianides and Ball's (2008), the result of the study suggested that only 26 % of the fifth graders were readily to offer successful new cases as the set of mathematics arguments. It seems that the successful new cases they were more likely to provide were the equivalent fractions of 1/2 and 1.

Regarding the modes of argument as the second component of proof, the result of the study indicated that empirical induction from finite number of discrete cases for verifying the task and offering more than one counterexamples for refuting a false proposition were the most two popular forms of reasoning used by fifth graders. Only about 10% of the fifth graders accepted that a single counterexample is sufficient to refute a false statement. This finding of the study becomes a literature of Reid and Knipping's (2010) reviewing that many secondary students do not accept counterexamples as refutation. The level of fifth graders' proof was characterised at Balacheff's (1988) first level, naïve empiricism.

In addition, making reference to an authority (the teacher) was also popular method of verification of the fifth graders. The result is consistent with Reid's study (1995). The result also showed that almost fifth graders' accepted the truth of "$\sim(P \rightarrow Q) \equiv Q \rightarrow \sim P$".

With respect to the third component of proof, the result of the study displayed that word description and it combines with pictorial representation were regular two representations of modes of argument used by the fifth graders.

The study suggested that knowledge of students' mathematics concepts and teachers' knowledge of mathematics embedded in the conjecturing tasks were two essential factors of affecting fifth graders' proofs and proving. Without solid mathematical concepts underpinning the arguments, it is impossible for students to produce logical proofs. The episode reported in the study showed that the teacher did not recognize whether Wei's statements were true or false. It leaded to the teacher's uncertainly to when and where to terminate students' discussions. The finding of the study supported Stylianides and Ball's (2008) claim. Unless the teacher had a good understanding of general rules of the comparison of fractions and sound knowledge of proof, we cannot expect that she was able to effectively promote proving. Helping teachers to assist students in engaging activities of valid proofs and proving is the purpose of the further study.

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