Lin, P. J. , Tsai, W. H. (2013). An alternative approach of teaching “axioms of equations” in elementary school. *Paper presented at the 5th Classroom Teaching Research for All Students Conference*. June 22 – 25, 2013, Fuzhou, Fujian, China.

AN ALTERNATIVE APPROACH OF TEACHING “AXIOMS OF EQUATIONS” IN ELEMENTARY SCHOOL

Pi-Jen Lin[1], Wen-Huan Tsai[2]

[(1)(2)]National Hsinchu University of Education

linpj@mail.nhcue.edu.tw, tsai@mail.nhcue.edu.tw

This study was to implement an alternative approach of the integration of SMART interactive white board and magic board into axioms of equations in an elementary classroom and to examine its effect on helping students promoting to relation meaning from operating meaning of equal sign. 32 students and 27 students from two sixth grade classrooms were assigned to experiment group and control group. Students in the two groups were conducted an instrument of axioms of equations for pre-test and post-test. The data collected for the study included: lesson plan, students' solutions gathered from classroom teaching, video tapes of teaching. The virtual beam balance was the main tool for helping students understanding the relation meaning of equal sign corresponding to algebraic equations. The result indicates that the use of technical devices contributed to students' understanding the relation meaning of equal sign. It resulted into the improvement of solving multiple-steps of algebraic equations.

key words: SMART interactive white board, magic board, equal sign, axioms of equations.

INTRODUCTION

Before year 2000s, students generally begin the study of algebra about 13 to 15 years old. Since 2001, algebra is brought into one of five topics in the school curriculum at elementary level (MOE, 2001). The early learning of algebra in the curriculum highlights that variables, algebraic expressions, equations, and equation solving should be developed over the grades. For instance, in the elementary grades, students typically develop a notion of ( ) as a placeholder for a specific number, as in ( ) + 5 = 8. Students in the middle grades require to learn a notion of variable x as a unknown number in $x+5=8$, solving the value of x by inversing addition to subtraction. In the high grades, students learn to solve the equation $4x+5=17$ by the use of “axioms of equation”, such as $4x+5-5=17-5$, then 4x=16.

The complexity and cognitive demand of learning algebra suddenly increase, such that it leads to more difficulties students encounter in learning algebra than in learning arithmetic. In arithmetic learning, the notion of equality is perceived as a signal to “do something” (Behr, Erlwanger & Nichols, 1975). Students in the early grades do not see the equals sign as a symbol that expresses the relationship “is the same as” (Behr, et al., 1975). Falkner, Levi, & Carpenter (1999) reported that elementary graders solved "8+4=( )+5" with 12 or 17. They do not accept the statement $5+3=6+2$. They think that the right side of the equation should indicate the answer, that is $5+3=8$. In algebraic learning, students should be developed to view the equal signs as a symbol of equivalence. Equal sign expresses a relation is one of the keys when developing mathematical and particularly algebraic thinking.

However, according to several research studies (Saenz-Ludlow and Walgamuth, 1998; Behr, Erlwanger and Nichols, 1975; Falkner, et al., 1999), students from grade 1 to 6 show serious misconceptions about the meaning of the equal sign. As Carraher, Schliemann and Brizuela (2000) suggest that students' previous mathematics instruction seem to be one of the main causes of many of the students' difficulties in early learning algebra. Capraro and Capraro (2007) further examining the textbooks for students' preparation show that the equal sign is presented in Chinese books as a relational symbol of equivalence which contrasted with the presentation in U.S. books. The result implicitly reflected pedagogical issues as a possible basis for the disparate results. It suggests that teachers should provide appropriate contexts that emphasize "equivalence" for students to learn the equal sign.

In traditional teaching, some teachers might use the physical manipulates such as balance with pan scales or some teachers might use chalk-and-board to teach the axioms of equations. In the former approach, it is hardly access to a balance by finding out two different concrete materials putting on the left and right side. In addition, it is time consuming. In the latter approach, it is very convenient for instructor, but it is not meaningful learning for students. Thus, the technological devices will be considered to be used to integrate into the axioms of equations, as an alternative instructional approach. The technological devices include SMART white board and Magic board, denoted as SIWB+MB.

SMART interactive whiteboards are about the same size as a standard whiteboard but are connected through a USB cable to a computer. The surface of the board is touching sensitive. Magic Board provides a collection of virtual objects that elementary teachers usually use to present mathematics concepts. MB is a web-based virtual aids environment for teaching elementary mathematics (Yuan, 2005). We integrating the MB into SIWB as the environment of interactive learning for the study was rooted in the following benefits of the virtual aids: (1) the virtual aids have the potential to overcome some of the main drawbacks of physical manipulatives (Clements & Sarama, 2005; Moyer, Niezgoda, & Stanley , 2005; Yuan, Lee, & Wang, 2010) and (2) the virtual aids have the nature of interactivity, flexible representations, unlimited supply, and easy to clean.

In order to overcome the difficulties and to ease the transition from arithmetic to algebra, this study is designed to design an alternative instructional approach to implement into classroom practice. The axioms of equations include that if $ax+b=c$, then it is true for (1) addition property, $ax+b+b=c+b$; (2) subtraction property, $ax+b-b=c-b$; (3) multiplication property, $1/a \times (ax+b) = 1/a \times c$, where $a \neq 0$; and (4) division property, $(ax+b) \div a = c \div a$, where $a \neq 0$. The axioms of equations is a prerequisite for understanding the meaning of simplifying terms step by step of solving equations.

The purpose of the study was to explore the effect of the SIWB+MB integrated into instruction of axioms of equations on students' leaning. There were two research questions to be answered: (1) How were the technical devices SIWB+MB integrated into axioms of equations? (2) What was the effect of the alternative instructional approach on the effect of students' learning on axioms of equations?

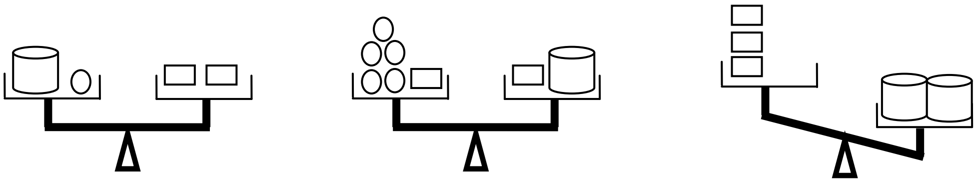
CONCEPTUAL FRAMEWROK

Three approaches of developing meaning of equations as an equivalence relation are adapted as the conceptual framework of the study.

Kieran's (1981) teaching experiment was designed to aid six 12- and 13-year-olds students in constructing meaning for algebraic equations of the type $ax \pm b = cx \pm d$. Kieran's experiment of teaching focused on extending the use of equal sign to include multiple operations on both sides, initially with one operation on each side, such as $2 \times 6 = 4 \times 3$ and $2 \times 6 = 10 + 2$. They then went on to construct equalities with multiple operations on each side. In the experiment, students seem in general to be quite comfortable with equality statements containing multiple operations on both sides when inserting the "answer" between both sides (i.e., $5 \times 3 = 15 = 10 + 5$). Students justified them in terms of both sides being equal because they had the same value. The equality symbol was being seen at this stage more as a relational symbol than as a "doing something signal". The right side, by this time, did not have to contain the answer, but rather could be some expressions that had the same value as the left side.

Bell, Malone, and Taylor's (1987) approach was a problem-solving one in which three classes of 14-year-old students are led to construct equations for problems as follows. Students are given the problem of 3 piles of rocks: A, B, C where B has 2 more than A and C has 4 times as many rocks as A. The total number of rocks is 14. Their task is to find the number of rocks in each pile using x and to do the problem in 3 "different ways"—i.e. using the x in three different positions (p.18). They report that all students start with pile A as 2and x, giving $x + 2$ and $4x$ for the other two piles. They write $x + x + 2 + 4x = 14$, so that $x = 2$. The second way, with pile B as x, students write $x - 2$ and $4x - 2$ for pile A and C, respectively. None use bracket $4(x - 2)$ for pile C. The resulting equation $x - 2 + x + 4x - 2 = 14$, does not get the same solution as before and consequently learn to a discussion on the need for brackets. The final way, with pile C as x, giving $1/4x$ and $2+ 1/4x$ for pile B and A, respectively. Most students write as $x \div 4 + 2 + x \div 4 + x = 14$.

Greens and Findell (1999) suggest that for learning equality to be effective, teachers need to provide students the opportunities to experiment with solving problems in which balance is maintained while modification is being made. Modifications include adding or subtracting the same amount to both sides of a balanced scale, multiplying or dividing both sides by the same positive factor, and making substitutions with equal amounts. Students need experience with balance and the different types of modification techniques in preparation for solving equations. They suggest that this experience can be provided with algebraic reasoning problems like the one involving pan scales in figure 1.



(a) (b) (c)

Figure 1: Pan scales

In each pan of the scales, there are collections of objects. Students have to analyze the contents of each pan, compare collections of objects, and deduce relationships about the masses of the objects. The inferred information may then be used to determine which object(s) should be placed in the pan on the left of the unbalanced scale to achieve balance.

Methodology

Research Design

The research design of this study was a quasi-experiment design. The quasi-experiment design includes an experiment group and a control group. Both groups received same an instrument for assessing the effect of treatment on pre-test and pos-test. Wen-Wen was the instructor of the experiment group and Li-Li was the instructor of the control group. 34 sixth-grade students consisted of 18 males and 14 females in experiment group and 27 sixth-grade students consisted of 14 males and 13 females.

Wen-Wen carried out the alternative approach of technology integrated into mathematics instruction. Wen-Wen and Li-Li were two of the six teachers participating in a teacher professional development program that was designed to enhance teachers' knowledge for mathematics teaching. Wen-Wen had been particitpated in the professional program, while Li-Li was the first year participation of the professional program. Wen-Wen gradually adopted learner-centered approach, but Li-Li was on the beginning way. Li-Li was used to take the traditional instruction and her classroom did not set up the environment of interactive white board.

Prior to teaching, Wen-Wen has analyzed instructional activities covered in textbook, conducting pre-test for students, and observed Jing-Jing's instruction in which made the first intervention of integrating the technology into mathematics instruction. Jing-Jing was one of the participants involving in the teacher professional program. All participants participated in a collective discussion for reflecting on Jing-Jing's teaching immediately right after classroom observation. After discussing, Wen-Wen took the comments from the group teachers of the professional program to Jing-Jing's into account in their re-designed lessons of the same topic in same grade. Reference to Jing-Jing's teaching, Wen-Wen integrated the SIWB and MB into axioms of equations.

Treatment: Integrating SIWB+MB into Teaching

Five sessions with 40 minutes were designed. Task 1 was for students to experience a balance of a beam by playing a game on the use of technical devices including SIWB and MB. Task 2 and 3 were designed to develop students' understanding the meaning of subtraction axiom of equations. The addition axiom of equation was processed in a transverse way. Task 4 was designed to develop students' understanding the meaning of division axiom of equations. The multiplication axiom of equation was processed in a transverse way. The fifth session, not the focus of the paper, was solving more than algebraic equations by using the axioms of equations.

The common flow of Wen-Wen's teaching was: (1) she proposed students a problem to solve; (2) students solved the problem individually; (3) she selected several students' solutions and sequenced the order of them for discussion; (4) The students whose solutions were selected were invited to explain their thinking individually in the whole class discussion; and (5) she summarized the main ideas of the discussion. Thus, Wen-Wen's classroom was full of the atmosphere of the interaction between the instructor and students.

Instruction of Control Group

Li-Li was the instructor of the control group. She was used to following the activities of the textbook. Five sessions related to axioms of equations with 40 minutes were engaged in the lesson. Although the actitives of the textbook also showed the pictures of beam balance for helping students understanding the meaning of axioms of equations, she did not appreciate the use of manipulative tool. Instead, she asked students to read the text of the textbook and interpreted it to students. She did not utilize the manipulative tool and the devices of SMART interactive white board and magic board. She stated that she was tired of the use of the manipulative tool of real beam in the instruction of axioms of equations, because it hardly kept a balance with real objects on the pan scales of both left and right sides of the beam.

Without the help of the mainipulative aids or technical devices, Li-Li's instruction was replaced by the chalk-and-board. Li-Li's speech occupied most of the time in a session. The flow of her instruction of axioms of equations was: (1) She asked students to read the text of the textbook; (2) Li-Li explained the meaning of the text by using one or two examples and demonstrated a solution in one way for a problem to be solved. (3) Li-Li gave students one or two problems to practice; and (4) Li-Li summarize the main idea of the lesson. She was not only a problem poser but also a problem solver. She was a speaker and her students were listener throughout the entire session in her teaching.

Data Collection and Analysis

The data collected for the study consisted of lesson plan, students' solutions gathered from two classrooms, video tapes of teaching, pre- and post-test for students on the axioms of equations.

The pre-test was conducted a week before the first session of the lesson, while the post test was conducted a month away from the pre-test. An instrument consisting of six items with 12 sub-items in total conducted in pre-test were to understand if students had the relation meaning of equal sign. Item 1, 5 and 6 were presented by a picture of a beam balance, while item 2, 3 and 4 were to present by a symbolic representation.

RESULTS

Students' Prior Knowledge of Learning Axioms of Equations

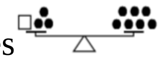
Table 1 was the correct percentages of students from the both groups performing on the pre-test. The t-test was utilized to test the significance of the score of pretest between the experiment group and control group. $t=1.53$, $df=57$, $p=.66>.05$, the data shows that students in both groups had a similar prior knowledge, since there was no significant difference between the two groups. Prior to teaching, students in both groups performed better on the items in which were presented with a beam balance than those of the items presented as an equation.

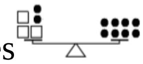
Table 1: The correct percentage of pretest of the experiment group (N=32) and control group (N=27)

Pictorial representation	Symbolic representation

						operating process					result	
item	1	5(1)	5(2)	6(1)	6(2)	2(1)	2(2)	3(1)	3(2)	4	2(3)	3(3)
Exp	69	47	91	38	88	13	56	9	56	50	75	75
Con	70	63	85	56	89	15	41	15	56	56	70	67

There were only 47% and 38% of the sixth grade students in the experiment group answered correctly to the item 5(1) and item 6(1), respectively. The two items were given as follows.

Item 5: A balance beam means that the weights of the objects in both sides  are the same.(1) If the weight of a「□」is x_{g}, a「●」is 1_{g}, what do you represent the balance with a mathematics symbols? (2) What are the weight of a「□」？

Item 6: A balance beam means that the weights of the objects in both sides  are the same. (1) If the weight of a「□」is x_{g}, a「●」is 1_{g}, what do you represent the balance with a mathematics symbols? (2) What are the weight of a「□」？

Only 13% and 9% of the sixth grade students in the experiment group answered correctly on the item 2(1) and 3(1), respectively. Likewise, only 15% of the students in control group succeeded in solving both the item 2(1) and 3(1). In experiment group, 56% of the sixth grade students answered correctly on both item 2(2) and item 3(2). Likewise, only 14% and 56% of the students in the control group succeed in solving the item 2(2) and item 3(2). The item 2 and item 3 were displayed as follows.

There was an interesting finding that students who were either in the experiment group or the control group performed on the item 2(3) were better than on the item 2(1) and item 2(2). Likewise, the performance on the item 3(3) was better than on the item 3(1) or item 3(2).

Table 2: Correct percentage of students in both groups performing on the item 2 and 3

Item 2	$x + 248 = 450$ $x + 248 -$ ((1)) $= 450 -$ ((2)) $x =$ ((3))	% of correct Exp. Con.	 (1) 13% (2) 56% (3) 75% (1) 15% (2) 41% (3) 70%
Item 3	$x - 405 = 395$ $x - 405 +$ ((1)) $= 395 +$ ((2)) $x =$ ((3))	Exp. Con.	(1) 9% (2) 56% (3) 75% (1) 15% (2) 56% (3) 67%

The data of the pre-test in Table 2 suggested that students had the difficulty with the understanding of the relation meaning of equal sign.

Warm up: playing beam game

After setting up the SIWB and MB, the instructor, Wen-Wen, put an apple on the pan scale at left side of the beam, Yi-Yi, one of students was invited to come to the front of the classroom and put something on the right side to be balanced. Yi-Yi used the way of try-and-error several times because of without knowing the weight of a tomato. Finally, the beam was in a balance with an apple on the left side and two 100g of counterpoises on the right side. The game kept playing in pairs lasting for about 3 minutes. The pairs students sometimes added, took away, double or triples, or half or third objects on the beam. Wen-Wen ended up the beam game with a statement for summary:"The axioms of equation were defined as the beam kept balance if one added and removed, multiplied, and divided same weights of objects."

Steps of teaching axioms of equations

Step 1: Helping students in transforming from a virtual beam in a balance to mathematical expression

The instruction of the definition of axioms of equations was followed by the instruction of transformation between virtual beam and abstract mathematical expressions. The balance beam was in a balance when a corm on the right side and an apple and a 50g-counterpoise on the left side of the beam. Students were asked to write the mathematical equation for the beam balance, as shown in Figure 2. Most of the students were able to represent the beam balance with $A+50=C$.

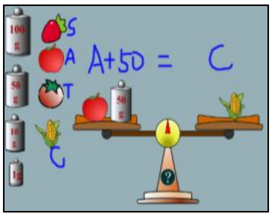


Figure 2: from virtual beam to mathematical expression

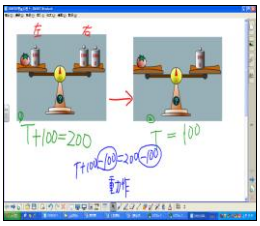


Figure 3: Subtraction axiom of equation

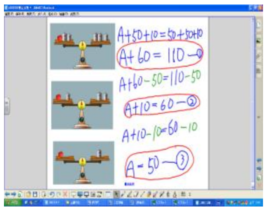


Figure 4: Subtraction axiom of equations

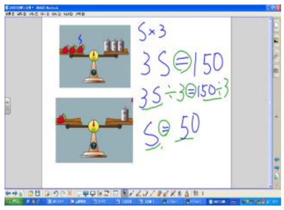


Figure 5: Division axiom of equations

Step 2: Helping students in writing down a mathematics equation to represent the action of "removing off" from a beam.

After getting more familiar with the transformation between virtual beam and mathematical expressions, students were asked to find the weight of an object by moving in and out from the pan scales of both side of the beam. Students were asked to write down the each mathematical equation corresponding to the each step of moving in and out. Wen-Wen asked students to follow three directions. First, jot down the original balance with a mathematical expression, such as $T+100=200$, Second, jot down a mathematics expression to represent the weight of a counterpoise you moved away from the beam, such as $T+100-100=200-100$. Finally, jot down a mathematics expression to represent a new balance, such as $T=100$.

Wen-Wen helped students to make connection of virtual world (virtual beam) and mathematics world (mathematics equations). For instance, the original balance turning into the new balance corresponded to from a mathematical equation T＋100＝200 (original balance) turning into T＝100 (new balance). She also helped students to make sense in operating mathematics equations in mathematics world. For instance , T＋100＝200→ T＝100 was resulted from T＋100－100＝200－100 indicated by the action of removing off a 100g-counterpoise.

Step 3: Helping students in resolving a multiple-steps equation by operating on a virtual beam.

Wen-Wen gave two more problems to figure out the weight of an apple and the weight of a strawberry by using the virtual beam balance, as shown in Figure 4 and Figure 5. The original balance of the beam shown on the first picture of Figure 4, were represented by a student Ting-Ting as A＋50＋10＝50＋50＋10 or A＋60＝110. In order to find the weight of an apple, first step, Ting-Ting removed off a 50g-counterpoise on both sides of the beam corresponding to a mathematics equation A＋60－50＝110－50 and got a second balance corresponding to a mathematics equation A＋10＝60. Second step, Ting-Ting continually removed off a 10g-counterpoise on both sides of the beam corresponding to a mathematics equation A＋10－10＝60－10 and got a third balance corresponding to a mathematics equation A＝50. For solving the problem, three balances occurred were marked by circles, indicated in Figure 4.

Effect of the Alternative Instructional Approach

The percentage of students performing successfully in each item on post-test was on Table 3.

Table 3: The correct percentage of each item on the posttest in both groups

	Pictorial representation					Symbolic representation						
						operating process					result	
item	1	5(1)	5(2)	6(1)	6(2)	2(1)	2(2)	3(1)	3(2)	4	2(3)	3(3)
Exp	91	88	91	81	100	91	94	91	94	88	100	97
Con	74	81	96	81	96	81	81	81	81	81	96	96

The result indicates that there were more students in the experiment group performing successfully on the symbolic representation representing the operating process then those students in the control group. We further examined if students performed on the items of the construct of representation on the posttest better than on the pretest. Only those t-values of the sub-constructs examined by t-test were achieved significantly listed on Table 4.

Table 4 suggests that the mean of experiment group and control group were 11.03 and 10.30, respectively. $t=1.67$, $df=57$, $p=.02 <.05$. The results indicate that students had similar prior knowledge ahead of learning the axioms of equations, through the treatment of the experiment, the use of technology integral into the axioms of equations contributed to students' achievement on the items relevant with axioms of equations.

Table 4: The significant differences of the sub-constructs between both groups

Sub-constructs		Pre-test					Post-test				
		$\bar{M}$	SD	t	df	p	$\bar{M}$	SD	t	df	p
Symb-operating	Exp.	1.81	.82	-2.48	57	.71	4.56	.80	1.80	57	.05*
	Con.	2.33	.78				4.07	1.27			
Symbol representation	Exp.	3.41	.84	1.28		.06	6.53	.92	1.81		.05*
	Con.	3.15	.91				6.00	1.33			
Total scores	Exp.	6.75	1.34	1.53		.66	11.03	1.40	1.67		.02*
	Con.	6.22	1.28				10.30	1.98			

*P<.05

Table 4 shows that students in experiment group had lower mean of scores on the pretest than students in control group, but there was no significant different. t=-2.48, df=57, p=.71>.05. Through the implementation of the use of technical devices on the axioms of equations, students in experiment group had significant higher mean of scores on the post-test than students in control group. t=1.80, df=57, p=.05, <.05. The result indicates that the use of SIWB+ MB significantly contributed to the learning of transformation from the operation of beam balance into the symbolic expression.

Conclusions

It is concluded that the use of technical devices including SMRT interactive white board and Magic board integrated into the instruction of axioms of equations contributed to promoting students understanding the meaning of equal sign to “relation” from “get into”. The improvement of students’ understanding the meaning of equal sign to “relation” corresponding to the axioms of equations attributing to the technical devices was rooted in the following six reasons.

First, the manipulation of virtual beam and objects (e.g. apples, pineapples, counterpoises) collected from Magic board displayed on the interactive white board inspired students’ involvement in the mathematical learning, since the virtual objects were quite similar with physical objects. Second, the virtual beam overcome the main drawback of a physical balanced scale that the modifications were difficult to adjust the same weight of physical objects in real world on both left and right sides of pan scales. Thus, the use of virtual objects manipulated on interactive white board was readily reached to accurate. Third, compared to the preparation of physical objects, the teacher in the experiment group did not need to spend too much extra time and energy to prepare the devices of the technology. Fourth, the virtual objects played on the interactive white board inspired students’ interaction with the instructor and peers. Two students in a pair were interestedly challenged in the modification on the beam to be a balance on both sides of the pan scales. Fifth, the devices of SIWB+MB were easy to trace and retrieve what students and the instructor have accomplished in a few minutes ago. Finally, the technical devices were flexible representations, unlimited supply, and easy to clean. The finding of the study was consistent with the previous studies on the use of virtual manipulative (Clements & Sarama, 2005; Moyer, Niezgoda, & Stanley , 2005; Yuan, Lee, & Wang, 2010)

Additional information

This article is based on the work supported by the National Science Council under grant no. NSC- 99-2521-S-134-005-MY3.Any opinions, findings, conclusions, or recommendations expressed are ours and do not necessarily reflect the views of the National Science Council.

References

Bell, A., Malone, J. A., & Taylor, P. C. (1987). *Algebra—an exploratory teaching experiment*. Nottingham, England: Shell Centre for Mathematical Education.

Behr, M. J., Erlwanger, S., & Nichols, E. (1975). *How children view the equals sentences*. (ERIC document Reproduction Service No. ED144802)

Carraher D., Schliemann A. D. &Brizuela B. M., (2000). Early Algebra, Early Arithmetic: Treating Operations as Functions. Paper presented in the Plenary Presentation at PMENA XXII , Tucson, AZ, October 7-10, 2000.

Clements, D. H., & Sarama, J. (2005). Young children and technology: What is appropriate? In W. J. Masalski (ed), *Technology-supported mathematics learning environments* (sixty-seventh yearbook), pp. 51-73. Reston, VA: NCTM.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999).Children's understanding of equality: a foundation for algebra, *Teaching Children Mathematics*, *6*, 232-236.

Greenes, C. Findell, C. (1999). Developing students' algebraic reasoning abilities. In Lee V. Stiff & Frances, R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12: 1999Yearbook* (pp.127-137). Reston: The National Council of Teachers of Mathematics.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317-326.

Ministry of Education (2001). Grade 1-9 Temporary Curriculum for Elementary and Junior High School: Mathematics. Taipei, Taiwan: The Author.

Moyer, P. S., Niezgoda, D., & Stanley, J. (2005).Young children's use of virtual manipulatives and other forms of mathematics representations. In P. C. Elliott (Eds.), *Technology-supported mathematics learning environments* (Sixty-seven yearbook). Reston, VA: NCTM.

Saenz-Ludlow, A. & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol, *Educational Studies in Mathematics 35*, 153-187.

Yuan, Y., Lee, C. Y., & Wang, C. H. (2010).A comparison study of polyominoes explorations in a physical and virtual manipulative environment. *Journal of Computer Assisted Learning*, 26(4), 307-316.