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A Task Design for Conjecturing in Primary Classroom Contexts

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The purpose of the study was to design tasks for conjecturing to support students engaging in the activities of proving in primary classroom contexts. A task involving perimeter and area of figures as an example was designed on the basis of nine principles of designing for conjecturing and proving suggested in F. L. Lin's team work (Lin, et al., 2012). Twenty seven third grade students' written work was collected for the main data of the study. It was found that the task was characterized as four features in accordance with students' justification, which were analyzed by three components of proofs suggested by Stylianides and Ball (2008) .

Key Words: mathematical conjecturing, proving, task design, primary,

Introduction

Mathematical tasks are recognized as the tool for teachers to shape a teaching design and for students to develop, utilize and understand a certain concept (Stein, Grover, Henningsen, 1996). Thus, mathematical instruction is generally organized and delivered through students' activities on mathematical tasks. Nevertheless, students respond to mathematical tasks very differently, depending on the structure and demands shaped by tasks enacted by teachers. This indicates that to reach high quality of mathematics instruction, mathematical tasks play a crucial role. Thus, selecting and designing appropriate tasks is essential to the success of teaching mathematics (Doyle, 1988; Stein & Lane, 1996).

Doyle (1983) defined academic tasks as (a) the products that students are to formulate, such as the answers to a set of questions; (b) the operations that are to be used to generate the product, such as classifying examples of a concept; and (c) the "givens" or resources available to students while they are generating a product. Doyle (1988) further argues that tasks with different cognitive demands are likely to induce different kinds of learning. According to this definition, tasks can vary not only with respect to mathematics content but also with respect to the cognitive processes involved in working on them. Only worthwhile tasks offer students the opportunity to extend what they know and stimulate their learning. Tasks that require students to solve complex problems can be considered to be cognitively demanding tasks. In contrast, cognitively undemanding tasks are those that give less opportunity for the students to engage in high-level cognitive processes. The conjecturing tasks require high cognitive demands because they are involved in three components: a set of true statements, valid modes of argumentation, and appropriate representation of modes of argumentation (Stylianides, 2007). Therefore, it is worthwhile to develop teachers' knowledge of conjecturing tasks design for enhancing students' proofs.

Moreover, proof is a vehicle to enhance students' understanding of mathematics concepts and promote mathematical proficiency and reasoning (Hanna, 2000). Proving is an important means of exploring in mathematics. Research shows that engagement in proving can support students to explore why things work in mathematics and explain their disagreements in meaningful ways, thus providing them with a solid basis for conceptual understanding (Stylianides, 2007). The previous studies suggest that students should have early and appropriate opportunities to incorporate proof into their mathematical learning (Ko, 2010; Kilpatrick, Swafford & Findell, 2001;Stylianides, 2007).

Conceptual Framework for Designing Conjecturing Tasks

Conjecturing and proving can promote mathematical thinking and launch mathematical inquiry; therefore, tasks of conjecturing and proving should be designed to be embedded into any grade level of classrooms (Kilpatrick, et al., 2001). The tasks of both conjecturing and proving are involved in students' conjecturing, but F. L. Lin and his colleagues distinguished the distinction of the principles for conjecturing tasks design from those for proving task design (Lin, et al., 2012). The four principles for designing conjecturing tasks and the five principles for designing proving tasks are considered as the conceptual framework of the study, since they are achieved to two fundamental functions: relating to the learners' roles or hypothetical learning trajectories and the practical function of easily evaluating.

In the tasks for conjecturing, students are asked not only to generate conjectures according to the given information which could be ill-defined, but also search for proofs to verify or justify whether the conjectures they made are true or not. They further suggest that the designing efficient tasks for conjecturing should consider the provision of opportunities to: (1) observation, (2) construct, (3) transform, and (4) reflect. The observation-based conjecturing refers to activities that involve purposeful or systematic focus on specific cases in order to make a generalization about the cases. The construction is a principle that encourages students to construct new knowledge based on prior knowledge which may lead to conjectures. The transformation design principle means that the task gives students the opportunities to generate conjectures by transforming given algorithms or formula. The conjectures by transformation may lead students to incorrect or meaningless statements. Thus, the reflection principle is essential to design the tasks for conjecturing.

The tasks for proving can promote proofs and proving. The type of tasks for proving is characterized as to ask students to justify whether the given statement in the existing conjectures is true. The false or true statement is given by the instructor's. Lin and his colleagues suggest that the tasks for proving have different difficulties, depending on the sources of conjectures are either from instructor's or students' themselves (Lin, et al., 2012). In the case of conjecture given by the instructor, students may feel it is not necessary to prove it, because they accept the truth based on either epistemic values (Duval, 1998) or instructor's academic authority (Lin &Tsai, 2012a; Reid, 1995). In the case of conjecture given by students themselves, they may have unclear distinction between conjecturing and proving. For instance, they may consider empirical arguments as deductive proof (Stylianides and Stylianides, 2009).

The five principles for designing proving tasks suggested by Lin's team have to do with modes of argumentation and its representations. They include: classifying mathematics statements, expressing arguments in several modes, changing roles in a task, defining efficient and necessary proof, and creating and sharing proof. The first principle is to provide the opportunity for students understanding six possible types of statement between universal and existential statements. It could be always true, sometimes true, and never true for true or false statement, respectively. The second principle is to provide the opportunity for students to understand various modes of argumentation to be used appropriately for different types of statement, such as counter examples, supportive examples, deductive and inductive proof via various representations. The third designing principle promotes the opportunities of switching the role of students as an instructor or an evaluator of the proposed justification from traditionally being seen as the learner. The fourth principle provides students the opportunities to be aware of the sufficiency and necessity for a legitimate proof. The final principle is to provide students the opportunity of creating a proof and then present it to the whole class in public for verifying its validity and truth.

A statement, existing in the tasks for proving, proposed by the instructor can be logically true or false. The existing empirical studies suggest that students at primary or secondary level do not accept counterexamples as refutation, rather, they offer more than one counterexamples for refuting a false statement (Reid & Knipping, 2010). Student even at high school level or undergraduates still have difficulty with mathematical proof in school mathematics (Lin &Tsai, 2012a; Ko, 2010). Some of them accept the truth of empirical induction from finite number of discrete cases for verifying a true statement. Besides, students or future teachers also perform better on false statement than true statement (Lin &Tsai, 2012b). The different difficulties may result from the nature of proving, because the verification of a false statement is easier than a true statement in that one counterexample is enough to refute a false statement. A true statement needs to be verified by inductive or deductive reasoning. The processes of proving a true statement demand rigor and complex argumentation.

On the basis of literature review, it could be a good start for the beginners at primary level for the study to learn proofs with a false statement instead of a true statement for designing proving tasks. However, it is new experience and knowledge that the task design for conjecturing are conducted by the primary teachers involving in the study, because their students have little experience with what conjecturing and proving looks like. Thus, the purpose of the study is intended to support teachers to design various conjecturing tasks in line with the mathematics contents to be taught in primary classroom contexts for student exploring the activities of proving. The paper describes not only the development of the task design but also the effect of conjecturing task on enhancing students' engaging in the activity of proving in the context of the relationship between perimeter and area in two figures. The task referred to in the study was the activities or artifacts such as teaching aids exploring in classroom contexts.

Method

Participants and Context

The task involved in the study was designed by one of the teachers who participated in the first year of a three-year project that was designed to help teachers to create conjecturing tasks for engaging students in the activity of proving. Hence, the tasks design for students engaging in valid proofs is a new experience and novice learning for the teachers, but they were mutually supported in the professional team consisting of six teachers and two researchers, the authors of the paper.

Twenty-seven third grade students in the class have separately learned the concepts of perimeter and area of a figure before engaging the task. They engaged in several activities of conjecturing in the first semester. The task was conducted in the second semester, so that some of the students had slight experience of finding out a counter example to refute a false statement.

They were grouped heterogeneously in groups of 4 or 5. After given the task, the students first worked independently and jotted down their judgement and verification on B4 paper; then they came together in groups to compare their solutions, and finally they shared their arguments to the whole class. The lesson was videotaped throughout the entire class. Each student's written work was collected throughout the whole year.

Designing Conjecturing Tasks for Students Engaging in the Activities of Proving

The task designed by the teacher was to ask students to make a conjecture and verify whether it is true. The statement is that "*In any two figures, if the area of one figure is bigger than the other, then its perimeter of the figure is greater than the other, too. Do you agree? Why? Show your work on the grid paper.* " The task for conjecturing is initiated from a false statement. The task design was on the basis of the four principles of the task for conjecturing and five principles of the task for proving, suggested by F. L. Lin's team work (Lin, et al., 2012), since it is potential to launch the following activities for students engaging in conjecturing and proving: (1) The task provided students an opportunity to engage in *observation* through finding out a pair of two figures and making a generalisation about the cases; (2) The task provided students an opportunity to engage in *construction*. For instance, to solve the task, students needed to create two figures with different areas but same perimeter; (3) The task encouraged students an opportunity to *transform* prior knowledge of the perimeter and area of an irregular figure by counting the number of small squares on the grid paper; and (4) The task provided students an opportunity for *reflection*. For instance, "*Show your work*" as part of the task is to ask students to explain why they believe their conjecture is true for the given condition.

In addition, the task was also characterised as the following features: (1) The task promoted students classifying various statements, such as, the statement of same area in two figures result in same perimeter; (2) The task had the potential to require students to express same argument in several representations of modes of argument. The counter examples for refuting the false statement can be expressed by either "*bigger area but smaller/same perimeter*" or "*same area but greater perimeter*"; (3) The task engaging in the classroom provided the students as the role of an evaluator for the justification proposed by them; (4) After the false statement was proved, the students were asked to modify it and rephrase it as a true statement; and (5) The task engaged in the classroom context provided students an opportunity with creating and sharing their own proofs.

Data Collection and Analysis

The result session was aligned to the Stylianides and Ball's (2008) three components of proof by using students' written work. Taking the consideration of students as mathematical learners, proof suggested by them can be defined as a set of *accepted* statements, *known* modes of argumentation, and *accessible* modes of argument representation to a classroom community. The three components of proofs as the framework of analyzing the data collected for the study, students' written solutions were first split into two piles and then a pile was assigned to each group of the two groups consisting of six school teachers studying in a master program. Afterwards, they took turns to review the other pile for increasing the validity and reliability of analysis.

Results

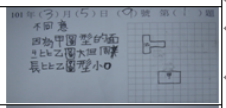
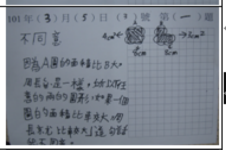
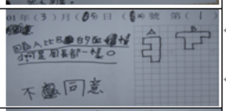
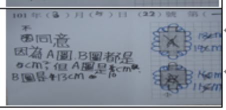
The Set of Statements Accepted by the Third Graders

After the task was explored, 8 (30%) students made incorrect judgment by accepting the teacher's statement, while 19 (70%) students judged correctly and verified successfully by finding out a pair of figures to reject the conjecture given by the teacher.

The set of statements referred to the statements accepted by the classroom community. Once they figured out a pair of figures, various accepted set of statements as part of their arguments were generated by 13 students, such as "*bigger area but smaller perimeter*", "*bigger area but same perimeter*", "*smaller area but same perimeter*", or "*same area but greater/smaller perimeter*". For instance, Ming, Jenny, Ron, and Huei, as the examples, successfully showed their accepted statements by their classmates, as shown in Table 1. Thus, possessing various types of statements for students' classification is the first feature of the task.

In addition, the statement "*The bigger area in one figure, it is not necessary to be greater in perimeter*." that 6 students used was another type of statement. Starting from the condition "*bigger area in one figure than the other*" given in the instructor's statement was the most common statements accepted by those who were in favor of the conjecture (in total, 68%, 13 out of 19).

Table 1: The Set of Statements Accepted by the Third Graders.

Students / Frequen.	Various accepted statements	Examples of the set of statements proposed by students: English version	Examples of the set of statements proposed by students: Chinese version
Ming / 7	Bigger area but smaller perimeter	Disagree. Because the area of figure A is bigger than B, while the perimeter of figure A is smaller than B.	
Jenny / 2	Bigger area but same perimeter	Disagree. Because the area of figure A is bigger than B, while their perimeters are same. Thus, I do not agree with the statement "*In any two figures, the bigger area of a figure has greater perimeter*".	
Ron / 1	Smaller area but same perimeter	Disagree. Because the area of figure A is bigger than B, but their perimeters are same in length.	
Huei / 2	Same area but greater/smaller perimeter	Disagree. Because the area of figure A and B are 5 cm^2, but the perimeter of figure A and B are 10 cm and 13 cm, respectively.	

Modes of Argument Known by the Third Graders

Modes of argument are the ways of verifying or justifying a statement. Overall, the mode of argument for the task used mostly by the third graders was the use of counterexample. They seemly knew that one counterexample is sufficient to refute a false statement. Their modes of argument for this task were not like other tasks that students are used to offer more than one counterexamples for refuting a false statement (Lin & Tsai, 2012a). Thus, the task provided the best opportunity for promoting students' understanding on proving that a single counterexample is sufficient to refute a false statement. This is the second feature of the task.

For verifying or refuting the statement given by the teacher, the students needed to find out a pair of figures such that one area in one figure is bigger/smaller

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than the other. It is followed by observing the relation of their perimeters. To fulfil the work, students needed clear distinction between the two concepts: area and perimeter. Sometimes, they needed to attempt several times, as shown in Figure 1. The Figure 1 displayed that the task was not only providing students an opportunity for exploring the activity of conjecturing and proving but also for clarifying students' confusion of perimeter with area. This is the third feature of the task.

Figure 1: Student' Tries for Finding Out a Pair of Two Figures

Representations of Modes of Argument Accessible by the Third Graders

The representations of modes of argument are the forms of expression for communicating with the classroom community. This conjecture task involving the relationship of perimeter and area, word expressions with figures was the most popular form of argument accessible by the third graders. As shown in Table 2. Eighteen (67%) students verified the statement by drawing at least an irregular figure in the pair of figures on the grid paper. It seems that third graders readily draw an irregular shape on the grid paper. As a consequence, it made students successful in conjecturing and justifying the statement. This is the fourth feature of the task.

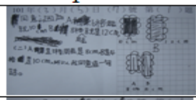
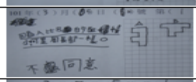
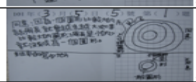
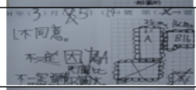
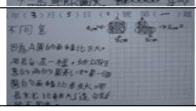
Shapes of a pair of figures	Frequencies	Examples
Two rectangles	9	
Two irregular shapes	8	
Two circles	1	
A rectangle and an irregular shape	7	
A square and an irregular shape	2	

Figure 2: Student' Representation of Figures

Conclusions and Discussions

The quality of task design was evaluated by students' justification on the basis of the three components of mathematical arguments, suggested by Stylianides and Ball (2008). It is said that the task maintained high cognitive demands while it was implemented into a third grade classroom, since the task was characterized as the following four features.

(1) The task provided students new experience that there were various types of statement instead of a single type for a counterexample to refute a false statement. It has to do with the first component of proofs: a set of statements. The students have developed knowledge of proofs for refuting a false statement by unique type of statement for a counterexample from prior tasks. This feature meets the first principle of task designing for proving suggested by F. L. Lin's team (Lin, et al., 2012).

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(2) The second feature was that the task provided the best opportunity for promoting students' understanding of a single case as a counterexample to be sufficient to refute a false statement. This feature is documented from the second component of proofs suggested by Stylianides and Ball (2008). The task was beneficial for developing students' knowledge of the way of refuting a false statement. This was different from previous studies in which most of the students at primary or secondary level were used to utilize more than one counterexamples to refute a false statement (Lin &Tsai, 2012a; Reid & Knipping, 2010).

(3) The task, which provided students an opportunity not only engaging the activity of conjecturing and proving but also clarifying students' confusion of perimeter with area, was characterized as the third feature. The feature of clarifying students' misconception or confusion between concepts is not on the list of task design principles suggested in F. L. Lin et al.'s work. The result indicated that the task could be a powerful instructional approach via conjecturing for clear understanding on perimeter and area. However, the effect of conjecturing on learning the relationship of perimeter and area needs further study in the future.

(4) The final feature was that the provision of grid paper as part of the task has potential to make students' successful in conjecturing and proving. The pictorial representation in several modes of argument was matched with the second principle of task design for conjecturing suggested in F. L. Lin et al.'s work (Lin, et al., 2012).

The study suggested that knowledge of students' mathematics concepts and teachers' knowledge of proofs embedded in the conjecturing tasks were two essential factors affecting the quality of proving exploring in the conjecturing activity. Without solid mathematical concepts underpinning the arguments, it is impossible for students to produce logical proofs. The study also suggested that the provision of false statements instead of true statements made it more possible for students to learn successfully on acquiring the knowledge of conjecturing and proving.

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